# Adding and Subtracting Fractions By Multiplying

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Abstract

This paper proposes a new way to do and teach adding and subtracting fractions with unlike denominators. It is based upon the current multiplication method. It introduces the story of the purpose and origin of the method in non-technical language, then goes into ten examples in non-technical language, remarks including technical proof and the method's reception, and six miscellaneous applications ranging from arithmetic to advanced algebra and back to arithmetic. There are 14 pages total.

Target Audience:

1. Those involved in teaching middle-grade students fractions regardless of profession;
2. Adolescents and adults who would like a *MUCH* easier way to add and subtract fractions.

## *Introduction*:

Every educator who has had to teach adding and subtracting fractions with unlike denominators[1] knows how nettlesome the topic is. To add and subtract fractions, common denominators are required, and the fractions have to be converted. I taught high school mathematics for two years. In the 1986-7 school year I was in fourth grade[2] and was myself on the receiving end of this instruction and facing such math problems. I especially dreaded this homework because I knew that it would be both intensely difficult and time consuming.

I learned to add and subtract fractions with unlike denominators by

1. seeking the lowest number all denominators in the involved fractions would evenly divide,
2. making that the new common denominator for all the fractions,
3. in each fraction, dividing the new denominator by the original denominator, taking the result, and multiplying the original numerator[1] by it,
4. making the results for all fractions the new numerators,
5. adding or subtracting numerators, leaving common denominator alone, then reducing if needed.

For me the hardest part was finding the lowest number that the two denominators divided evenly and then doing the divisions and conversions. In sixth grade I decided that the most efficient way for me to add and subtract fractions with unlike denominators when I did not quickly recognize the Least Common Denominator = LCD was to simply multiply both denominators together to get a common denominator, multiply each numerator by the other denominator, calculate, and reduce. This often resulted in needlessly cumbersome fractions to calculate and reduce. Anytime both the numerator and denominator are divisible by the same number besides 1, the fraction should usually be reduced. Such cumbersome fractions often required heavy reduction.

Even as a future mathematics teacher, adding and subtracting fractions with unlike denominators still challenged me in the last year of my elementary education. It is no surprise that in my experience teaching at the high school and college levels I have seen many students simply unable to cope with such calculations. Experiences dealing with teaching or reinforcing skills in such calculations have been paralleled by many teachers who address such calculations.

As I stated earlier, step 1 was hardest for me when I was learning, and step 3 could be difficult when dividing large denominators. When I began teaching, many of my students had impasses with these steps; these were the most common problems for students that I saw instructing the old method, and because of these difficulties, 1) `I can't do fractions' is a common partly-serious joke among adults, and 2) fractions cause severe anxiety in many students ranging from school-age to adulthood.

[1] The top number is the NUMERATOR and the bottom number is the DENOMINATOR.
[2] Starting fraction calculations has since been moved to fifth grade in Indiana.

Note: This is an unofficial/nonprofessional paper, so it has a slightly higher likelihood of unnoticed imperfections.

A related subject is how I learned to multiply fractions. Right after those sections on adding and subtracting fractions in fourth grade, I still remember what glad news it was to me when I saw that multiplying fractions was easier than adding or subtracting them. Common denominators were unnecessary, and at that time we just multiplied across and reduced. From elementary school until 2002 this was how I multiplied fractions. In spring of 2002 I had just completed teaching licensure at Marian College and was hired at South Ripley Jr./Sr. High School in Versailles, Indiana to cover a maternity leave for junior high mathematics. This was fifteen years after I first learned fraction multiplication. The students there were using a method that I had seen in college but did not yet use myself. I adopted this method to avoid discontinuity with the students' prior instruction. The method was a diagonal cancellation method, as follows:

*New:* $\frac{7}{10} x \frac{15}{28}$ ==>

$$\frac{{}^{1}\cancel{7}}{{}_{2}\cancel{10}} x \frac{\cancel{15}^{3}}{\cancel{28}_{4}} = \frac{3}{8} \quad \leftarrow \boxed{\begin{array}{l} 1x3 = 3 \\ 2x4 = 8 \end{array}}$$

*Old:* $\frac{7}{10} x \frac{15}{28} = \frac{7x15}{10x28} = \frac{105}{280}\frac{\div 5}{\div 5} = \frac{21}{56}\frac{\div 7}{\div 7} = \frac{3}{8}$

Results of straight multiplication | Reduce start here

We recognized that diagonals 7 and 28 were both evenly divisible by 7 and divided both by 7. We recognized that diagonals 10 and 15 were evenly divisible by 5 and divided both by 5. After this we multiplied straight across. This is the method being presented in textbooks at this time. After my time at South Ripley, I taught both methods to students and allowed them their choice during two years of teaching high school. Presently, I show community college students both methods initially for the comfort of students who were in school in the 1980's and before, but after that I vigorously advocate and use the new algorithm.

After I taught two years of high school mathematics, I spent the 2004-2005 school year substitute teaching predominantly elementary school, mostly in Switzerland County, Indiana, plus in spring of 2005 was promoted to a part-time supervision and instructional support position at an elementary school there. The following summer I took a summer job tutoring for a campus of a statewide community college, specifically the Ivy Tech Community College campus at Columbus, Indiana. My tutoring experience involved levels from general mathematics to calculus and included Intermediate Algebra Math 111. To add and subtract algebraic fractions, which are fractions with variables, with unlike denominators, a multiplication trick using the denominators was applied.[3] I suspected that there might be a similar trick for arithmetic fractions, but did not act on that suspicion until I got an adjunct instructor position that involved teaching a Basic Math Skills course. After teaching the standard way to convert and add/subtract fractions with unlike denominators with the usual struggle to a Math 040 class in fall of 2005, I recalled what I had thought about over the summer and decided to see if there was anything to it. That was my task that evening, and in a couple of hours I found that my suspicion was correct.

The textbook used by the Columbus campus of Ivy Tech Community College for Math 040 and Math 044 teaches multiplication and division of fractions before addition and subtraction.[4] This was reasonable as the former operations are simpler than the latter operations, and ideal practice for mathematics instruction is to progress from less difficult to more difficult material. The sequencing of the textbook was well suited for the five-step method that my investigation produced:

[3] As outlined in section 6.2 of Tobey, John and Jeffrey Slater. Intermediate Algebra. Pearson Prentice Hall, 2002. ISBN 0-13-032837-5.
[4] Tobey, John and Jeffrey Slater. Basic College Mathematics. Pearson Prentice Hall, 2005. ISBN 0-13-149057-5.

Presenting A New Method:

1. multiply each fraction by an artificial fraction having the other denominator over itself;
2. diagonally cancel the ***original left*** denominators and ***only*** the original denominators, never canceling the other pairs of denominators even if possible, and canceling both pairs of fractions by the same single number;
3. multiply straight across to finish the conversions into fractions with a common denominator, and do ***not*** reduce converted fractions -- they have been described as "expanded" and "inflated" fractions and are always reducible to their original unconverted states, defeating the purpose;
4. add or subtract the numerators as appropriate leaving common denominator untouched;
5. ***after*** calculation in step 4, reduce as needed.

This method avoids a need to recognize a common denominator before beginning conversions, which is a sticking point. Students do not need to scour their memories for the lowest number two denominators divide, nor do they need to make tables of multiples, nor do they need prime factorization. Rather, the common denominator is quickly calculated as are the corresponding numerators. I think this will make fractions ***much*** easier for most people. Allow me to demonstrate.

*Example 1*:

$$\frac{5}{12}+\frac{3}{8}$$

*The Method Presented Here*

Convert:

$$\frac{5}{12}\times\frac{8}{8} \implies \frac{5}{\cancel{12}_3}\times\frac{\cancel{8}^2}{8}=\frac{10}{24}$$

$$\frac{3}{8}\times\frac{12}{12} \implies \frac{3}{\cancel{8}_2}\times\frac{\cancel{12}^3}{12}=\frac{9}{24}$$

Cancel by 4.

Let the reader please notice that in the second conversion, the diagonals 3 and 12 could have been canceled. However, recall that step 2 in our algorithm excludes canceling diagonal pairs involving non-original denominators even if possible. Please notice also that neither converted fraction was reduced.

Calculate:

$$\frac{10}{24}+\frac{9}{24}=\frac{19}{24}$$

*The Standard Method Presently Used*

Before we can do anything, we need to find out the lowest number that both denominators 8 and 12 divide evenly. We could use prime factorization or tables of multiples; we opt for the latter:

*8*: 8, 16, 24, 32, 40, 48, 56,64
*12*: 12, 24

The Least Common Denominator is 24.
We can now do our conversions:

$$\frac{5}{12}=\frac{?}{24}\text{; } 12\overline{)24}=2,\ 5\times2=10\text{, so }\frac{10}{24}$$

$$\frac{3}{8}=\frac{?}{24}\text{; } 8\overline{)24}=3,\ 3\times3=9\text{, so }\frac{9}{24}$$

Calculate:

$$\frac{10}{24}+\frac{9}{24}=\frac{19}{24}.$$

There is no reduction possible in either case, so we have our final answer.

I tend to do calculations horizontally after conversion because this method is useful for three or more fractions and for negative fractions in algebra, and horizontal seems most conducive in both situations. *Example 7* will involve negative fractions, and *Example 10* will involve three fractions.

It might be important at this point in the new algorithm to ensure that students understand what exactly the original problem is. If students were multiplying fractions straight out, they would have wanted to cancel the 3 and the 12 in the second conversion. The original problem, however, is an addition or subtraction problem, and some students may need to be reminded to restrict how they multiply in this problem accordingly.

*Example 2*:

$$\frac{11}{12} - \frac{4}{15}$$

Most of us do not have multiples of 15 memorized and would not quickly recognize that the lowest number both 12 and 15 divide evenly is 60. This knowledge is not needed under the new algorithm.

Convert:

$$\frac{11}{12} \times \frac{15}{15} \Longrightarrow \frac{11}{{}_{4}\cancel{12}} \times \frac{\cancel{15}^{5}}{15} = \frac{55}{60}$$

$$\frac{4}{15} \times \frac{12}{12} \Longrightarrow \frac{4}{{}_{5}\cancel{15}} \times \frac{\cancel{12}^{4}}{12} = \frac{16}{60}$$

Cancel by 3.

Sidenote: The old method would have had us
1. trying to find the lowest number that 12 and 15 divide evenly, which is 60,
2. dividing 60 by 12, and
3. division 60 by 15

before converting the numerators. This new method avoids that necessity.

Let the reader again note that the diagonals 4 and 12 in the second fraction were not canceled even though they both could have been canceled by 4. Only original denominators were canceled. Also, neither converted fraction was reduced.

Calculate:

$$\frac{55}{60} - \frac{16}{60} = \frac{39}{60}$$

Reduce: 39's digits 3 and 9 add up to 12, which is divisible by 3, so 39 is divisible by 3. 60's digits 6 and 0 add up to 6, which is divisible by 3, and so 60 is also divisible by 3. Hence, both 39 and 60 are divisible by 3, so dividing the numerator 39 and the denominator 60 by 3 gives us:

$$\frac{13}{20}.$$

*Example 3*:

$$\frac{12}{35} + \frac{13}{25}$$

Most students would not recognize within three seconds that the lowest number both 35 and 25 divide evenly is 175.

Convert:

$$\frac{12}{35} \times \frac{25}{25} \Longrightarrow \frac{12}{{}_{7}\cancel{35}} \times \frac{\cancel{25}^{5}}{25} = \frac{60}{175}$$

$$\frac{13}{25} \times \frac{35}{35} \Longrightarrow \frac{13}{{}_{5}\cancel{25}} \times \frac{\cancel{35}^{7}}{35} = \frac{91}{175}$$

Cancel by 5.

Calculate:

$$\frac{60}{175} + \frac{91}{175} = \frac{151}{175}.$$

No reduction is possible, so we have our final answer.

This problem is above the degree that middle or elementary school students would typically see. However, this is the type of problem that college math courses might involve.

*Example 4*:

$$\frac{1}{4} + \frac{3}{8}$$

Many students would immediately recognize that 8 is the lowest number both 4 and 8 divide evenly. They might still, however, not immediately recognize how to convert the one-fourth to a fraction of 8. Unfortunately, many other students would be looking for the lowest number that both 4 and 8 divide but which is not one of them, which would be 16.

Convert:

$$\frac{1}{4} \text{ x } \frac{8}{8} \text{ ==> } \frac{1}{{}_{1}\cancel{4}} \text{ x } \frac{\cancel{8}^{2}}{8} = \frac{2}{8}$$

$$\frac{3}{8} \text{ x } \frac{4}{4} \text{ ==> } \frac{3}{{}_{2}\cancel{8}} \text{ x } \frac{\cancel{4}^{1}}{4} = \frac{3}{8}$$

Cancel by 4.

Calculate:

$$\frac{2}{8} + \frac{3}{8} = \frac{5}{8}.$$

No reduction is possible, so we have our final answer.

Please notice that even though we tried to convert three-eighths, we still ended up with three-eighths. Both fractions ended up having the Least Common Denominator before calculation.

We have studied several pairs of fractions whose denominators are divisible by a common number other than 1. Let us see what happens when the denominators involved are relatively prime, meaning that no other whole number besides 1 divides them both evenly.

*Example 5*:

$$\frac{3}{5} - \frac{2}{7}$$

Convert:

$$\frac{3}{5} \text{ x } \frac{7}{7} \text{ ==> } \frac{3}{5} \text{ x } \frac{7}{7} = \frac{21}{35}$$

$$\frac{2}{7} \text{ x } \frac{5}{5} \text{ ==> } \frac{2}{7} \text{ x } \frac{5}{5} = \frac{10}{35}$$

No cancellation.

No whole number other than 1 divides both 5 and 7, so no diagonal cancellation is possible. We simply multiply straight across to get our converted fractions, and of course we do not reduce.

Calculate:

$$\frac{21}{35} - \frac{10}{35} = \frac{11}{35}$$

There is no reduction possible.

*Example 6*:

$$7\,\frac{2}{9} - 4\,\frac{5}{6}$$

These are mixed numbers. What we will do is convert the fractional parts, and then put those back into the mixed numbers and calculate.

Convert:

$$\frac{2}{9} \text{ x } \frac{6}{6} \text{ ==> } \frac{2}{{}_{3}\cancel{9}} \text{ x } \frac{\cancel{6}^{2}}{6} = \frac{4}{18}$$

$$\frac{5}{6} \text{ x } \frac{9}{9} \text{ ==> } \frac{5}{{}_{2}\cancel{6}} \text{ x } \frac{\cancel{9}^{3}}{9} = \frac{15}{18}$$

Cancel by 3.

Calculate:

Original Problem: $7\frac{2}{9}-4\frac{5}{6}$ ==> Converted Fractions: $7\frac{4}{18}-4\frac{15}{18}$

$$\begin{array}{rl} 7\frac{4}{18} & \text{==> } 6\,\frac{18}{18}+\frac{4}{18} = 6\frac{22}{18} \\ -\;4\frac{15}{18} & \phantom{\text{==> } 6\,\frac{18}{18}+\frac{4}{18}} = 4\,\frac{15}{18} \\ \hline & \phantom{\text{==> } 6\,\frac{18}{18}+\frac{4}{18} =\;} 2\,\frac{7}{18} \end{array}$$

Alternative Way of Thinking:

$$\begin{array}{rl} \overset{6}{\cancel{7}}\,\frac{4+18}{18} & = 6\frac{22}{18} \\ -\;4\frac{15}{18} & = 4\frac{15}{18} \\ \hline & \;\; 2\frac{7}{18} \end{array}$$

There is no reduction possible. We have our final answer.

This method of conversion is useful beyond arithmetic. The next example will be algebra.

*Example 7*:

$$-\frac{2}{25}-\frac{11}{20}$$

Convert:

$$\frac{2}{25}\times\frac{20}{20} \text{ ==> } \frac{2}{{}_5\cancel{25}}\times\frac{\cancel{20}^{4}}{20}=\frac{8}{100}$$

$$\frac{11}{20}\times\frac{25}{25} \text{ ==> } \frac{11}{{}_4\cancel{20}}\times\frac{\cancel{25}^{5}}{25}=\frac{55}{100}$$

Cancel by 5.

With negative fractions, which are less than zero, I recommend converting the `unsigned' part just like one would a positive fraction, and after conversion return it to its negative status.

Calculate:

$$-\frac{8}{100}-\frac{55}{100} \text{ ==> } \frac{-8}{100}-\left(\frac{55}{100}\right) \text{ ==> } \frac{-8}{100}+\left(-\frac{55}{100}\right) \text{ ==> } \frac{-8}{100}+\left(\frac{-55}{100}\right)=\frac{-63}{100}=-\frac{63}{100}.$$

With signed fractions, I suggest students move any negative signs to the numerators. Then I suggest that they do the calculations for the numerators just like signed integers. I refer to positive numbers as "gains" and negative numbers as "losses." For subtraction problems, I use what was introduced to me as "Leave-Change-Opposite."[5] You *leave* the first number as is, *change* the operation to addition, and replace the second number with its *opposite*. Hence, -8-55 meaning -8-($^+$55) becomes -8+(-55), and a "loss of 8" plus a "loss of 55" totals to a "loss of 63" = -63. We return the negative sign to the middle for the final form of the fraction, which does not reduce.

*Example 8*:

$$\frac{41}{30}+\frac{25}{18}$$

Convert:

$$\frac{41}{30}\times\frac{18}{18} \text{ ==> } \frac{41}{{}_5\cancel{30}}\times\frac{\cancel{18}^{3}}{18}=\frac{123}{90}$$

$$\frac{25}{18}\times\frac{30}{30} \text{ ==> } \frac{25}{{}_3\cancel{18}}\times\frac{\cancel{30}^{5}}{30}=\frac{125}{90}$$

Cancel by 6.

Calculate:

[5] Presented to former high school students of mine by David Diel, presently a teacher of mathematics at Clay City High School at Clay City, Indiana.

$$\frac{123}{90} + \frac{125}{90} = \frac{248}{90}.$$

Reduce: as both the top and the bottom are even, we know that the fraction reduces under division by 2. This gives us a final fraction of $\frac{124}{45}$.

*Example 9*:

$\frac{7}{2} + 4$ Remember that 4 can be written as a fraction $\frac{4}{1}$.

Convert:

$$\frac{7}{2} x \frac{1}{1} = \frac{7}{2}$$

$$4 = \frac{4}{1}: \quad \frac{4}{1} x \frac{2}{2} = \frac{8}{2}$$

No cancellation.

Calculate:

$$\frac{7}{2} + \frac{8}{2} = \frac{15}{2}.$$

There is no reduction possible.

Finally, this method works also with more than two fractions. The following is our last example.

*Example 10*:

$$\frac{5}{8} + \frac{1}{6} - \frac{7}{12}$$

We will work this problem two ways. One way will be to add the first two fractions and then afterward subtract the third, and the other way will be to convert all three at once and then calculate.

*Way 1*

Convert Part One:

$$\frac{5}{8} \times \frac{6}{6} \Longrightarrow \frac{5}{{}_4\cancel{8}} \times \frac{\cancel{6}^3}{6} = \frac{15}{24}$$

$$\frac{1}{6} \times \frac{8}{8} \Longrightarrow \frac{1}{{}_3\cancel{6}} \times \frac{\cancel{8}^4}{8} = \frac{4}{24}$$

Cancel by 2.

Calculate Part One:

$$\frac{15}{24} + \frac{4}{24} = \frac{19}{24}$$

Convert Part Two:

$$\frac{19}{24} \times \frac{12}{12} \Longrightarrow \frac{19}{{}_4\cancel{24}} \times \frac{\cancel{12}^2}{12} = \frac{38}{48}$$

$$\frac{7}{12} \times \frac{24}{24} \Longrightarrow \frac{7}{{}_2\cancel{12}} \times \frac{\cancel{24}^4}{24} = \frac{28}{48}$$

Cancel by 6. We could have used 12.

Here it was decided to cancel diagonally using 6 even though we could have used 12. Many children would not recognize 24 as a multiple of 12, but are more likely to recognize 12 and 24

*Way 2*

Convert:

$$\frac{5}{{}_4\cancel{8}} x \frac{\cancel{6}^3}{6} = \frac{15}{24} \Longrightarrow \frac{15}{{}_{2\,4}\cancel{24}} x \frac{\cancel{12}^{2\,1}}{12} = \frac{15}{24}$$

$$\frac{1}{{}_1\cancel{6}} x \frac{\cancel{12}^2}{12} = \frac{2}{12} \Longrightarrow \frac{2}{{}_3\cancel{12}} x \frac{\cancel{8}^2}{8} = \frac{4}{24}$$

$$\frac{7}{{}_3\cancel{12}} x \frac{\cancel{8}^2}{8} = \frac{14}{24} \Longrightarrow \frac{14}{{}_4\cancel{24}} x \frac{\cancel{6}^1}{6} = \frac{14}{24}$$

For the second part of the first conversion, we first canceled 24 and 12 by familiar number 6, then canceled those results by 2. We could have canceled by 12 at once; what we did has the same effect.

What we did was to multiply each fraction by **either** of the other denominators over itself and cancel ***only*** the ***left*** denominator. Then, after multiplying across with no further cancellation, we took the result and multiplied by the third denominator over itself, and again canceled ***only*** the ***left*** denominator. We finished the conversion by multiplying straight across.

as multiples of 6. As long as both fractions are canceled diagonally in multiplication by the same number, this method will still work.

Calculate Part Two:

$$\frac{38}{48} - \frac{28}{48} = \frac{10}{48}$$

Reduce: 10 and 48 are both divisible by 2, so dividing 10 and 48 by 2 reduces the fraction to $\frac{5}{24}$. ∢✶✶✶

This shows us that even if we do not cancel by the highest number possible, we can still get the correct result if we cancel both original denominators by the same number and reduce after calculation.

By using this way, we avoid having to ensure that three fractions all convert to fractions with the same single denominator. However, this way of doing the calculation might take longer.

Calculate:

$$\frac{15}{24} + \frac{4}{24} - \frac{14}{24} = \frac{5}{24}$$ ∢✶✶✶

This fraction does not reduce.

This way of doing this calculation involves only a single conversion and calculation sequence. However, it may not always work out that you cancel by a high enough number or enough numbers that all three conversions get canceled to the same denominator. On the first conversion, had we not gone beyond the cancellation by 6, we would have been stuck with $\frac{30}{48}$ instead of $\frac{15}{24}$ and a need to either:

1. convert the other fractions of 24 into fractions of 48,
2. see to reduce $\frac{30}{48}$ to correct fraction of 24, or
3. notice we had a fraction of 24 before the second part of the conversion, and go back.

This could get complicated.

Both ways of applying the method work and each has pro's and con's over the other. The reader has the prerogative as to what to do on a case-by-case basis when faced with three or more fractions. If one way does not seem to work well for the person solving, s/he can utilize the other way.

*Remarks*:

It is important to notice that in every pair of fractions to be added or subtracted, both fractions in the pair were canceled diagonally in the multiplication by the exact same number without exception. This is proper method.

This method can be proven. Let us add two fractions $\frac{a}{bn} + \frac{c}{dn}$ where $n$ is the product of all factors shared by the denominators and equals 1 if both denominators are relatively prime, $b$ is the product of those factors in the first denominator not shared by the second denominator and $d$ is the product of those factors in the second denominator not shared by the first denominator. Of course, neither $b$, $d$, nor $n$ equal zero.

| Standard Method | Presented Method |
| --- | --- |
| A common denominator for $\frac{a}{bn} + \frac{c}{dn}$ would be $bdn$. To convert the first fraction, $bdn$ would be divided by $bn$, leaving $d$, and that $d$ would be multiplied by $a$ for a new numerator $ad$: $bdn \div bn = d$ and $a \cdot d = ad$. | Convert: First fraction $\frac{a}{bn} \cdot \left(\frac{dn}{dn}\right) = \frac{adn}{bd \cdot n \cdot n} = \frac{ad\cancel{n}}{bd \cdot \cancel{n} \cdot n} = \frac{ad}{bdn}$ The last step coming after $n$ is factored out of the numerator and denominator of the second expression above and canceled. In this algorithm, we define the $n$ canceled in the denominator as the $n$ from the left fraction. |

| Standard Method | Presented Method |
|---|---|
| To convert the second fraction, $bdn$ would be divided by $dn$, leaving $b$, to be multiplied by $c$ for a new numerator $bc$:<br>$bdn \div dn = b$ and $b \cdot c = bc$.<br><br>The resulting converted fractions and subsequent calculation yield:<br>$\frac{ad}{bdn} + \frac{bc}{bdn} = \frac{(ad + bc)}{bdn}$. | Second fraction $\frac{c}{dn} \cdot \left(\frac{bn}{bn}\right) = \frac{bcn}{bd \cdot n \cdot n} = \frac{bc\cancel{n}}{bd \cdot \cancel{n} \cdot n} = \frac{bc}{bdn}$<br>the last step coming after $n$ is factored out of the numerator and denominator of the second expression above and canceled. Again, we define $n$ canceled in the denominator as $n$ from the left fraction.<br>Calculate:<br>$\frac{ad}{bdn} + \frac{bc}{bdn} = \frac{(ad + bc)}{bdn}$. |

The last equations under both methods are exactly the same. Hence, the methods lead to mathematically equivalent results. As subtraction is really the addition of the additive inverse, this proof is valid for subtraction also.

One great advantage of this method is that it does not require students to know the lowest number two denominators divide evenly before beginning calculation; determining this becomes part of the actual calculation. One pedagogical implication of this method is to make the practice of introducing fraction multiplication before fraction addition/subtraction to be the standard procedure. As mathematics instruction typically advances from less complex to more complex, and multiplication of fractions is simpler than addition and subtraction of fractions, this makes sense. This is reflective of best practice for fractions only, and does not reflect how we handle the teaching of whole number operations and decimal operations.

After I worked this method out, I showed it to my fall 2005 Math 040 students at the Columbus campus of Ivy Tech Community College. This was the class session immediately following the original session over the topic. My teaching style is to demonstrate problems using multiple methods when possible. For example, in one topic of Math 044 the same single problem was done four different ways. After I showed this method to my Math 040 class, the students were very excited to see it, and in fact later mentioned that they wanted me to do every problem with this method first before redoing it the standard way. Since that time, classes that I have taught have insisted on seeing this method almost exclusively after being shown both methods.

*Miscellaneous Common Denominator Problem #1*

This method can also be used or modified for any type of problem needing the Least Common Denominator. For instance, compare with <, >, or = without converting to decimals:

$$\frac{7}{12} \_\_\_ \frac{11}{18}.$$

Convert:

$$\frac{7}{12} \times \frac{18}{18} = \frac{7}{{}_{2}\cancel{12}} \times \frac{\cancel{18}^{3}}{18} = \frac{21}{36}$$

$$\frac{11}{18} \times \frac{12}{12} = \frac{11}{{}_{3}\cancel{18}} \times \frac{\cancel{12}^{2}}{12} = \frac{22}{36}$$

Cancel by 6.

Resolution of question: $\frac{21}{36} < \frac{22}{36}$ so $\frac{7}{12} < \frac{11}{18}$. As a reminder of where the inequality sign points, a normal hungry alligator's opened mouth would face the greater meal, in this case $\frac{11}{18}$, and the pointed tail would point to the lesser meal which s/he would decide against.

*Miscellaneous Common Denominator Problem #2*

The Least Common Denominator of multiple fractions can be determined using a method resembling this. In basic algebra, we can clear the following equation of fractions using the LCD [6]:

$$\frac{2x}{15} + \frac{x+3}{12} = \frac{3(x-2)}{8} - \frac{1}{3} + 1$$

Here is what can be done to find the lowest number evenly divisible by 15, 12, 8, and 3:

$\frac{15}{15} x \frac{12}{12}$ cancel left denominator, then times $\frac{8}{8}$ and repeat, then do the same times $\frac{3}{3}$ and repeat:

$$\frac{15}{\underset{5}{\cancel{15}}} x \frac{\overset{4}{\cancel{12}}}{12} = \frac{60}{60} \implies \frac{60}{\underset{15}{\cancel{60}}} x \frac{\overset{2}{\cancel{8}}}{8} = \frac{120}{120} \implies \frac{120}{\underset{40}{\cancel{120}}} x \frac{\overset{1}{\cancel{3}}}{3} = \frac{120}{120}$$

$\boxed{\text{Cancel by 3.}}$ $\boxed{\text{Cancel by 4.}}$ $\boxed{\text{Cancel by 3.}}$

*This tool is based on and resembles our fraction conversion method, but it is* **NOT** *the same thing and has different use: the LCD. Normal Least Common Multiples can also be found this way.*

This gives an LCD of 120, as would other orders of this process. When the whole equation -- all terms on both sides -- is multiplied by that LCD of 120, we get this:

$$(120)\frac{2x}{15} + (120)\frac{x+3}{12} = (120)\frac{3(x-2)}{8} - (120)\frac{1}{3} + (120)1$$

In multiplication, whole numbers can be canceled against denominators. We do these cancellations:

$$(\overset{8}{\cancel{120}})\frac{2x}{\cancel{15}} + (\overset{10}{\cancel{120}})\frac{x+3}{\cancel{12}} = (\overset{15}{\cancel{120}})\frac{3(x-2)}{\cancel{8}} - (\overset{40}{\cancel{120}})\frac{1}{\cancel{3}} + (120)1$$

$16x + 10(x+3) = 45(x-2) - 40 + 120$ and going on from there we get:

$26x + 30 = 45x - 10$

$30 = 19x - 10$

$40 = 19x$

$x = \frac{40}{19}$.

These last steps are: distributive property and combining like terms, then canceling the smaller *26x* to zero by subtracting it from each side, then canceling off the *-10* term from the side with the variable by adding back 10 on both sides, and then dividing off the coefficient *19* leaving the solution as an improper fraction because 19 does not divide 40 evenly.

*Miscellaneous Common Denominator Problem #3*

A similar application of the Least Common Denominator in algebra to what we saw just above can be done in systems of simultaneous equations with two variables LATER in algebra courses than what we saw in *Miscellaneous Problem #2*. Let us solve the following system of equations:

$$7x - 6y = -4$$
$$-4x + 9y = 19.$$

On a coordinate graph, the solution to this system is where the lines of the equations intersect. At

[6] Procedure using LCD outlined in section 2.4 of Tobey, John and Jeffrey Slater. Beginning Algebra. Pearson Prentice Hall, 2006. ISBN 0-13-148287-4.

this intersection, the $x$-coordinates and $y$-coordinates of both lines are the same. Hence, the value of $y$ at the intersection will be the same for both lines. We can use a modification of the substitution method to solve this system. We would solve each equation as a literal equation for $y$ and then set the results equal to each other and solve numerically for $x$.

$$\begin{array}{rcl} 7x - 6y &=& -4 \\ -7x & & -7x \\ \hline -6y &=& -4 - 7x \\ y &=& \dfrac{-4-7x}{-6} \end{array} \qquad \begin{array}{rcl} -4x + 9y &=& 19 \\ +4x & & +4x \\ \hline 9y &=& 19 + 4x \\ y &=& \dfrac{19+4x}{9} \end{array}$$

I move the negative of the denominator to the top, changing all signs top and bottom:

$$y = \frac{4+7x}{6}$$

Why? $\dfrac{-4-7x}{-6} = \dfrac{(-4-7x)}{-6} = \dfrac{-(-4-7x)}{6} = \dfrac{4+7x}{6}$

Both equations are solved for $y$, so now we set them equal to each other:

$$\frac{4+7x}{6} = \frac{19+4x}{9}$$

Note: This can be compared with *Example 6* where we also had 6 and 9 for denominators. Here we are not converting but rather are seeking the LCD.

We are going to clear the fractions with the LCD just as we did in *Miscellaneous Problem #2*. First, we find the LCD if in doubt, just as we did in *Miscellaneous Problem #2*:

$\dfrac{6}{6} x \dfrac{9}{9}$ ==> $\dfrac{6}{\cancel{6}_2} x \dfrac{\cancel{9}^3}{9} = \dfrac{18}{18}$. Hence, the LCD is 18. We now multiply both sides of the equation by 18:

$18\left(\dfrac{4+7x}{6}\right) = \left(\dfrac{19+4x}{9}\right)18$ ==> $\overset{3}{\cancel{18}}\left(\dfrac{4+7x}{\cancel{6}}\right) = \left(\dfrac{19+4x}{\cancel{9}}\right)\overset{2}{\cancel{18}}$ after canceling the denominators against the 18's. This gives us: $3(4+7x) = 2(19+4x) \Rightarrow\rightarrow 12 + 21x = 38 + 8x$

$$\begin{array}{rcl} 12 + 21x &=& 38 + 8x \\ -8x & & -8x \\ \hline 12 + 13x &=& 38 \\ -12 & & -12 \\ \hline \dfrac{\cancel{13}x}{\cancel{13}} &=& \dfrac{26}{13} \\ x &=& 2. \end{array} \quad ⋞★★★$$

Technical note: Using cross-multiply method of proportions would have been valid but yielded:

$9(4+7x) = 6(19+4x) \Rightarrow\rightarrow 36+63x = 114+24x$

with larger numbers to calculate than what we obtained from clearing the fractions via the LCD.

Plugging $x = 2$ into the second original equation gives us $-4(2) + 9y = 19$

$$\begin{array}{rcl} -8 + 9y &=& 19 \\ +8 & & +8 \\ \hline 9y &=& 27 \\ \dfrac{\cancel{9}y}{\cancel{9}} &=& \dfrac{27}{9} \\ y &=& 3. \end{array} \quad ⋞★★★$$

The second equation was chosen because it involved fewer minus/negative signs.

The solution is $x = 2$, $y = 3$ also written (2,3). This `double substitution' method might be a good alternative to normal substitution or elimination for solving systems of simultaneous equations.

### *Miscellaneous Common Denominator Problem #4—With Parallel Numeric Example*

We will be returning to arithmetic next problem, but the following method based on the Least Common Denominator procedure above would be appropriate for ***ADVANCED*** algebra. Among fully accredited colleges, this following problem represents material included around the middle or toward the end of 100-level mathematics courses in algebra. In this problem, we will *simplify* the following complex rational expression:

$$\frac{\dfrac{2x}{x+3}+\dfrac{3x-1}{x^2+5x+6}}{\dfrac{5}{4x}+8}$$

NOTE: This is upper-level algebra. If you are not well familiar with

1. polynomial operations, and
2. polynomial factoring,

this will reasonably make NO sense to you at all.

First, we need to factor everything that we can. Hence, $x^2+5x+6 = (x+3)(x+2)$, so the second fraction in the numerator is:

$$\frac{3x-1}{x^2+5x+6}=\frac{3x-1}{(x+3)(x+2)}$$

Presently, the author knows of two ways to simplify the original expression.[6] One is to simplify the top and bottom separately into single rational expressions, and then divide the top by the bottom, which means multiply the top by the reciprocal of the bottom.

The other way is to multiply all the terms on both top and bottom by the LCD of all algebraic fractions top and bottom. With the method proposed here of multiplying and canceling left denominators and converting, this might be the least cumbersome way to simplify. Observe.

First, find the LCD of x+3, (x+3)(x+2), and 4x:

$$\frac{x+3}{\cancel{x+3}}\bullet\frac{\cancel{(x+3)}(x+2)}{(x+3)(x+2)}=\frac{(x+3)(x+2)}{(x+3)(x+2)}\bullet\frac{4x}{4x}=\frac{4x(x+3)(x+2)}{4x(x+3)(x+2)} \leftarrow \text{LCD}$$

Multiply all by LCD:

$$\frac{\dfrac{2x}{\cancel{x+3}}(4x)\cancel{(x+3)}(x+2)+\dfrac{3x-1}{\cancel{(x+3)}\cancel{(x+2)}}(4x)\cancel{(x+3)}\cancel{(x+2)}}{\dfrac{5}{\cancel{4x}}\cancel{(4x)}(x+3)(x+2)+8(4x)(x+3)(x+2)}$$

yields the following much simpler expression and process:

$$\frac{2x(4x)(x+2)+(3x-1)(4x)}{5(x+3)(x+2)+8(4x)(x+3)(x+2)}=\frac{8x^3+16x^2+12x^2-4x}{5x^2+25x+30+32x^3+160x^2+192x}$$

$$=\frac{8x^3+28x^2-4x}{32x^3+165x^2+217x+30}=\frac{4x(2x^2+7x-1)}{32x^3+165x^2+217x+30}$$

*Similar Numeric Example:*

Calculate: $$\frac{\dfrac{3}{4}+\dfrac{7}{5}}{\dfrac{1}{6}+2}$$

First, find the LCD between 4, 5, and 6:

$$\frac{4}{4}x\frac{5}{5}=\frac{20}{20}$$ Nothing cancels, then

$$\frac{20}{\cancel{20}_{10}}x\frac{\cancel{6}^{3}}{6}=\frac{60}{60}$$ cancel ***left*** denominator.

LCD

Now, multiply everything top and bottom by LCD 60 applying the Distributive Property, then cancel diagonals and calculate:

$$\frac{\dfrac{3}{\cancel{4}}(\cancel{60}^{15})+\dfrac{7}{\cancel{5}}(\cancel{60}^{12})}{\dfrac{1}{\cancel{6}}(\cancel{60}^{10})+2(60)}$$

Same as multiplying by $\frac{60}{60}$.

This gives us:

$$\frac{45+84}{10+120}=\frac{129}{130}.$$

In Math-M 001 at Indiana University-Purdue University Columbus, all instructors are supposed to teach both methods, and the problems in that course are not as advanced as the one here. Between the two, I expected a better student response simplifying the top and bottom independently and then simplifying the resultant complex fraction. This is because I expected that finding the LCD of two fractions on top and the LCD of two fractions on bottom would seem simpler than trying to come up with the LCD of all fractions involved. To my surprise, however, my Math-M 001 class at IUPUC in spring 2006 unanimously told me that using the overall LCD seemed most expedient to them because of this method of finding the LCD. I considered this worth reporting. This problem is substantially more tedious than most textbook exercises of its type. THIS WAS OUR *LAST* APPEARANCE OF ALGEBRA; OUR NEXT AND FINAL MISCELLANEOUS APPLICATION WILL BE BACK IN THE REALM OF ARITHMETIC.

[7] These methods are both explained and given equal time in section 6.8 of Bittinger, Marvin. Introductory Algebra. 9th edition. Addison Wesley, 2003. Paperback student edition ISBN 0-201-74631-X.

*Miscellaneous Common Denominator Problem #5*

We now return to arithmetic. This modified method from *Miscellaneous Problem #2* which can produce the Least Common Denominator of multiple fractions can also be used in conjunction with the standard method currently taught and used in the schools and outlined on page 1. For instance, say we want to do $\frac{2}{3}+\frac{1}{4}-\frac{7}{8}+\frac{5}{6}$. Imagine that we do not know our multiplication tables well enough to know that the lowest number the denominators 3, 4, 8, and 6 divide evenly is 24. Here is how we could find the LCD without having our multiplication tables memorized or using prime factorization or making multiple tables:

$\frac{3}{3} x \frac{4}{4} = \frac{12}{12}$ ==> $\frac{12}{{}_3\cancel{12}} x \frac{\cancel{8}^2}{8} = \frac{24}{24}$ ==> $\frac{24}{{}_4\cancel{24}} x \frac{\cancel{6}^1}{6} = \frac{24}{24}$

Cannot cancel. Cancel by 4. Cancel by 6.

Put first two denominators over themselves to multiply, cancel *left* denominator if possible, multiply across, take result and set up next denominator, repeat for all.

Hence, the LCD is 24. Proceeding with the standard method as outlined on page 1 yields:

$\frac{2}{3} = \frac{\ }{24}$ $3\overline{)24} = 8$ and $2 \times 8 = 16$, so $\frac{2}{3} = \frac{16}{24}$

$\frac{1}{4} = \frac{\ }{24}$ $4\overline{)24} = 6$ and $1 \times 6 = 6$, so $\frac{1}{4} = \frac{6}{24}$

$\frac{7}{8} = \frac{\ }{24}$ $8\overline{)24} = 3$ and $7 \times 3 = 21$, so $\frac{7}{8} = \frac{21}{24}$

$\frac{5}{6} = \frac{\ }{24}$ $6\overline{)24} = 4$ and $5 \times 4 = 20$, so $\frac{5}{6} = \frac{20}{24}$

An alternative method: convert each fraction by setting it to be multiplied by the Least Common Denominator over itself, then cancel the *left* denominators dividing by themselves leaving "1," and then multiply to finish conversion. Notice that the division and multiplication calculations that occurred to our left also happened here.

$\frac{2}{{}_1\cancel{3}} x \frac{\cancel{24}^8}{24} = \frac{16}{24}$ $\frac{7}{{}_1\cancel{8}} x \frac{\cancel{24}^3}{24} = \frac{21}{24}$

$\frac{1}{{}_1\cancel{4}} x \frac{\cancel{24}^6}{24} = \frac{6}{24}$ $\frac{5}{{}_1\cancel{6}} x \frac{\cancel{24}^4}{24} = \frac{20}{24}$

The calculation would yield:

$\frac{16}{24}+\frac{6}{24}-\frac{21}{24}+\frac{20}{24} =$ 16+6-21+20 = 21: Leave denominator alone: ==> $\frac{21}{24}$ Both divisible by 3; do it: ==> $\frac{7}{8}$.

I showed two ways to handle more than two fractions in *Example 10*, but this may be best for some.

*Miscellaneous Common Denominator Problem #6*

The alternative conversion method of *Miscellaneous Common Denominator Problem #5* can be used for TWO OR MORE fractions. It will be used for this problem:

"Put in order from least to greatest without converting to decimals: $\frac{7}{10}, \frac{13}{20}, \frac{8}{15}, \frac{2}{3}$."

First, let us find the Least Common Denominator:

Third Denominator Fourth Denominator

$\frac{10}{{}_1\cancel{10}} \times \frac{\cancel{20}^2}{20} = \frac{20}{20}$

LCD of 10, 20

$\frac{20}{{}_4\cancel{20}} \times \frac{\cancel{15}^3}{15} = \frac{60}{60}$

LCD of 10, 20, 15

$\frac{60}{{}_{20}\cancel{60}} \times \frac{\cancel{3}^1}{3} = \frac{60}{60}$

The Least Common Denominator of the four fractions is 60.

Done properly, doing these denominators in any order should yield an LCD of 60. Also, done properly, doing any two denominators this way should yield the LCD between those two, and doing any three denominators this way should yield the LCD of those three.

Now that we have found the LCD, let us convert the fractions:

$$\frac{7}{10}\times\frac{60}{60} \Rightarrow \frac{7}{\cancel{10}_{1}}\times\frac{\cancel{60}^{6}}{60} = \frac{42}{60}$$

$$\frac{13}{20}\times\frac{60}{60} \Rightarrow \frac{13}{\cancel{20}_{1}}\times\frac{\cancel{60}^{3}}{60} = \frac{39}{60}$$

$$\frac{8}{15}\times\frac{60}{60} \Rightarrow \frac{8}{\cancel{15}_{1}}\times\frac{\cancel{60}^{4}}{60} = \frac{32}{60}$$

$$\frac{2}{3}\times\frac{60}{60} \Rightarrow \frac{2}{\cancel{3}_{1}}\times\frac{\cancel{60}^{20}}{60} = \frac{40}{60}$$

This same conversion method could be used to add and/or subtract these four fractions.

← It can be difficult to recognize the appropriate cancellation on these middle two IF we do not remember to always cancel by the original denominators. ←

All fractions are now converted. We place them in numerical order on basis of their numerators:

$$\frac{32}{60},\frac{39}{60},\frac{40}{60},\frac{42}{60}$$

$$\downarrow\quad\downarrow\quad\downarrow\quad\downarrow$$

$$\frac{8}{15},\frac{13}{20},\frac{2}{3},\frac{7}{10}.$$

Application Note: This is usually done by converting to decimals, which requires calculators to be time-efficient. Using this conversion method would often be a time-efficient way to do this without calculators.

*To close, I wish you the very best with fractions whether teaching or learning.*

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